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Stable orbits embedded in a chaotic attractor for a trapped ion interacting with a laser field

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Abstract. The interaction between a trapped ion and a resonant laser standing wave is studied under the analytical approximation. Periodic orbits next to the heteroclinic one and their stability conditions are derived from the Rayleigh perturbation method. Theoretical analysis reveals the stable periodic orbits to be embedded in the Melnikov chaotic attractor. The corresponding numerical results show that fitting control parameters into the stability conditions can control chaos in the system.

As a mesoscopic system, laser-cooled ions confined to a Paul trap have served as a simple model to investigate classical and quantum chaos [1–5]. Recently, there has been much interest in the interaction between a single-trapped ion and a resonant laser standing wave [6–10]. The Melnikov analysis [11], the Lyapunov characteristic exponent [12] and the numerical methods were used in these works. It was shown that chaos can occur in the system consisting of a single two-level ion, a laser field and the trap. The parameter region and conditions that lead to instability and chaotic motion were given analytically and numerically [10].

In this paper, we base the study of this topic on the Rayleigh perturbation. Under the secular approximation, we consider the small applied voltages to trap and the laser field in a standing-wave configuration such that the single two-level ion system obeys the Melnikov perturbed equation

$$\dot{\boldsymbol{x}} = \boldsymbol{h}_0(\boldsymbol{x}) + \varepsilon \boldsymbol{h}_1(\boldsymbol{x}, t) \qquad |\varepsilon| \ll 1. \tag{1}$$

By applying our perturbation technique [13, 14] to equation (1), we obtain the Rayleigh series solution near the heteroclinic orbit. It is proved that the perturbed periodic orbits are Lyapunov stable iff some relationships between physical parameters and initial conditions are satisfied. The stability conditions contain Melnikov's chaos criterion that implies the stable periodic solutions being embedded in a chaotic attractor. In order to control chaos, we must fit the control parameters into the stability conditions. The analytical results are in agreement with the corresponding numerical results.

For the considered ion system, the evolution of external and internal dynamics is dominated by the Melnikov perturbed equation (1) with [10, 15]

$$\boldsymbol{x} = (x, v) \qquad \boldsymbol{h}_0 = (v, -\Omega \sin x) \qquad \varepsilon \boldsymbol{h}_1 = (0, q h_1)$$
$$h_1 = 2\cos(2\tau)\left(x + \frac{\pi}{2} - \phi\right). \tag{2}$$

Here x and v represent the position and velocity of the ion centre-of-mass; Ω is proportional to the Rabi frequency, the energy recoil, and is inversely proportional to the square of the micromotion frequency ω; $q = \varepsilon$ depends on the specific geometry of the trap and the applied voltage; ϕ is the relative position between the centre of the trap and the laser standing wave; $\tau = \omega t/2$ denotes the dimensionless time. Substituting equation (2) into equation (1) yields the two-dimensional equations.

$$\dot{x} = v \qquad \dot{v} = -\Omega \sin x + q h_1(x, \tau) \tag{3}$$

where the overdot indicates derivative with respect to τ. Suppose that q is so small that the term proportional to it may be regarded as perturbation added to the unperturbed equations

$$\dot{x}_0 = v_0 \qquad \dot{v}_0 = -\Omega \sin x_0. \tag{4}$$

Applying the Rayleigh perturbation expansions

$$x = \sum_{i=0} q^i x_i \qquad v = \sum_{i=0} q^i \dot{x}_i \tag{5}$$

to equation (3) and equating the sum of ith-order terms to zero, we obtain the zeroth-order equations (4) and the ith-order equations

$$\dot{x}_i = v_i \qquad \dot{v}_i = -\Omega(\cos x_0)x_i + h_i(x_{i-1}, \tau) \qquad i = 1, 2, \ldots. \tag{6}$$

The unperturbed equations (4) possess the well known heteroclinic orbit [10]

$$x_0 = 2\arctan(\sinh\xi) \qquad v_0 = \pm 2\sqrt{\Omega}\,\mathrm{sech}\,\xi \tag{7a}$$
$$\xi = \pm(\sqrt{\Omega}\tau + C) \qquad C = Ar\sinh[\tan x_0(\tau_0)/2] \pm \sqrt{\Omega}\tau_0 \tag{7b}$$

where τ_0 is the initial time. Given equations (2) and (7), the Melkinov function simply becomes [10, 11]

$$\Delta(\tau_0) = \int_{-\infty}^{\infty} \boldsymbol{h}_0 \wedge \boldsymbol{h}_1 \exp\left[-\int \mathrm{Tr}\, D_x \boldsymbol{h}_0(s)\, \mathrm{d}s\right] \mathrm{d}\tau = \int_{-\infty}^{\infty} v_0 h_1\, \mathrm{d}\tau. \tag{8}$$

Previous work [10] calculated the integration (8) and pointed out that $\Delta(\tau_0)$ has simple zeros, indicating the existence of stochastic behaviour for the orbits whose initial conditions are sufficiently near the unperturbed heteroclinic orbit (7). The purpose of this paper is to derive such orbits from equations (5)–(7) and to obtain a detailed analytical criterion for controlling the Melnikov chaos.

By using our perturbation technique [13, 14], the general solutions of equations (6) can easily be constructed as

$$x_i = u_0 \int_{A_i}^{\tau} v_0 h_i\, \mathrm{d}\tau - v_0 \int_{B_i}^{\tau} u_0 h_i\, \mathrm{d}\tau \tag{9a}$$
$$v_i = \dot{u}_0 \int_{A_i}^{\tau} v_0 h_i\, \mathrm{d}\tau - \dot{v}_0 \int_{B_i}^{\tau} u_0 h_i\, \mathrm{d}\tau \qquad i = 1, 2, \ldots \tag{9b}$$

with arbitrary constants A_i and B_i which may be adjusted by the initial conditions. Here the function v_0 is given in equation (7) and u_0 has the form

$$u_0 = v_0 \int (v_0)^{-2}\, \mathrm{d}\tau = \tfrac{1}{4}\Omega^{-1}(\sinh\xi + \xi\,\mathrm{sech}\,\xi). \tag{10}$$

From equations (4) and (10) we have the equations

$$\ddot{v}_0 = -\Omega(\cos x_0)v_0 \qquad \ddot{u}_0 = -\Omega(\cos x_0)u_0 \qquad v_0\dot{u}_0 - u_0\dot{v}_0 = 1. \tag{11}$$

Given equations (11), it is straightforward to show that equations (9) are general solutions of equations (6). The properties of the solutions (9) are quite interesting. Obviously, these solutions possess periodicity, since h_1 includes the periodic function $\cos(2\tau)$. For small parameter q and finite τ_0, the initial conditions of the orbit (5) with equations (7) and (9) are certainly near the heteroclinic orbit. However, the ith-order corrected solutions (9) seem to be divergent as $\tau \to \pm\infty$, because u_0 and $\dot{u}_0$ tend to infinity at that time. This usually means the solutions (5) are Lyapunov unstable [16]. However, fortunately, the instability can be controlled by some necessary and sufficient conditions. That is to say, the solutions (5) with equations (7) and (9) are Lyapunov stable iff the coefficient functions of u_0 and $\dot{u}_0$ in equations (9) satisfy the conditions

$$F_{i\pm} = \lim_{\tau\to\pm\infty} \int_{A_i}^{\tau} v_0 h_i \,\mathrm{d}\tau = 0 \qquad i = 1, 2, \ldots. \tag{12}$$

The necessity of the conditions is evident, because of the divergence of u_0. Applying equations (12), (7), (10) and the l'Hospital rule to equations (9) and (6) results in the superior limits

$$\overline{\lim_{\tau\to\pm\infty}} x_i = \overline{\lim_{\tau\to\pm\infty}} h_i \qquad \overline{\lim_{\tau\to\pm\infty}} v_i = \overline{\lim_{\tau\to\pm\infty}} \dot{h}_i. \tag{13}$$

For the small parameter q and the finite h_i we therefore have

$$\|q^i \boldsymbol{x}_i\| = [(q^i x_i)^2 + (q^i v_i)^2]^{1/2} < \delta_i \qquad i = 1, 2, \ldots \tag{14}$$

at all times where δ_i are some small constants. This is just proof for the sufficiency of the stability conditions (12).

Then we see how the dependence of conditions (12) on the integration constants A_i leads to sensitivity of the orbits to the initial conditions. Setting (x', v') and (x'_i, v'_i) are another set of orbits (5) and (9) with the constants A'_i, B'_i and the same zeroth-order terms $(x'_0 = x_0)$, combining equations (5) with equations (9) and (7) we have

$$x - x' = \sum(x_i - x'_i) = u_0 \sum \left(\int_{A_i}^{\tau} v_0 h_i \,\mathrm{d}\tau - \int_{A'_i}^{\tau} v_0 h_i \,\mathrm{d}\tau \right)$$
$$-v_0 \sum \left(\int_{B_i}^{\tau} u_0 h_i \,\mathrm{d}\tau - \int_{B'_i}^{\tau} u_0 h_i \,\mathrm{d}\tau \right) = A u_0 - B v_0 \tag{15a}$$

$$v - v' = \sum(v_i - v'_i) = A\dot{u}_0 - B\dot{v}_0 \tag{15b}$$

with arbitrary constants A and B depending on the integral constants A_i, A'_i and B_i, B'_i respectively. The constants A and B will not equal zero unless $A_i = A'_i$ and $B_i = B'_i$ such that $x = x'$, $v = v'$. Some ith-order small differences between the initial conditions of (x, v) and (x', v') will lead to the different A_i, A'_i, B_i, B'_i and the nonzero A and B. Thus, at least one of the orbits (x, v) and (x', v') does not obey the conditions (12) if $A_i \neq A'_i$. Consequently, the differences (15) between the two orbits will tend to infinity as $\tau \to \pm\infty$, because of the infinite $u_0(\pm\infty)$ and $\dot{u}(\pm\infty)$. These clearly show that the orbit (5) sensitively depends on the initial conditions.

We know the sensitivity to initial conditions being the general character of chaos. Combining equation (8) with equations (12) we find $F_{1+} - F_{1-} = \Delta(\tau_0) = 0$. That is, the first-order stability conditions contain the Melnikov criterion for the onset of chaos. This means the stable orbit given by equations (5) with equations (7) and (9) is to be embedded in the Melnikov chaotic attractor. In view of the Perron–Lyapunov characteristic exponent [16], the largest Lyapunov exponent reads

$$\lambda(\boldsymbol{x}) = \overline{\lim_{\tau\to\infty}}(\tau^{-1}\ln\|\boldsymbol{x}\|)\begin{cases} = 0 & \text{for } A_i \text{ given by equations (12)} \\ > 0 & \text{for the other } A_i \end{cases} \tag{16}$$

with $\boldsymbol{x}$ being the orbit (5). Clearly, $\lambda > 0$ indicates the instability of the orbit $\boldsymbol{x}$ and the case $\lambda = 0$ is usually called the critical one. Thus equation (16) exhibits the sensitivity of the stability to the initial constants A_i again and points out the stable orbits of the critical case. The intersections of the stable and unstable orbits are determined by conditions (12). Therefore equations (12) are really the necessary and sufficient conditions for the onset of chaos. They supply a more detailed analytical criterion of chaos than the Melnikov-function technique.

Further, we explore how to control chaos through conditions (12) and (16). A normal technique [17] is to adjust the control parameters such that the Lyapunov exponent gets to zero. In general, from equation (16) we cannot immediately determine the Lyapunov exponent, since the initial constant A_i cannot be set experimentally. Only by adjusting the control parameters of the system to fit conditions (12), can we obtain the zero Lyapunov exponent and control the chaos in the experiments. Let us see the most important case of equations (12) with $i = 1$. Inserting h_1 and equations (7) with positive signs into equations (12), we get the first-order control conditions

$$F_{1\pm} = \lim_{\xi\to\pm\infty}\int_{A_1}^{\xi} \operatorname{sech}\xi \cos[2\sqrt{\Omega^{-1}}(\xi - C)][\tfrac{\pi}{2} - \phi + 2\arctan(\sinh\xi)]\,\mathrm{d}\xi = 0 \tag{17}$$

with $\xi = \sqrt{\Omega}\tau + C$. These are two non-integrable integrations which necessitate numerical calculations. In equations (17), the control parameters are the frequency Ω and position ϕ. The integration constants C and A_1 depend on the initial conditions which cannot be set experimentally. In order to fit any one of equations (17) by adjusting the control parameters, we first consider the effect of constant C. From $F_{1+} - F_{1-} = \Delta(\tau_0) = 0$ we calculate ϕ as a function of Ω numerically for (a) $C = 1$, (b) $C = 10$ and (c) $C = 100$. In the calculations, we have taken the upper and lower limits of the integration as $\xi = \pm 20$, since $\operatorname{sech}\xi = \operatorname{sech}(\pm 20) \approx 0$ in the integrand of equations (17). The numerical results are shown in figure 1. In this figure, the ϕ versus Ω curves become right lines as the values of C increase. At the case of right lines, it is easy to fit the control conditions by changing the values of Ω for different and great C. Taking large τ_0 or making $x_0(\tau_0)$ near to heteroclinic point $x_0(\tau_0) \approx \pi$, from equations (7*b*) we can obtain the great C. Fixed any great C, for example $C = 10$, numerical calculations from the first of equations (17) ($F_{1+} = 0$) leads the control curves to figure 2, where (a) $A_1 = 0$, (b) $A_1 = 2$ and (c) $A_1 = 4$. In figure 2 we see that changes of Ω values can easily fit the control curves, as our assertion. On the other hand, figure 2 shows that adjusting the parameter ϕ from 2 to 5 can also fit the control curves. Once any one of the control curves is reached, stable state of the system occurs and remains indefinitely such that chaos is effectively controlled [14, 17].

In conclusion, we have treated a trapped two-level ion interacting with a resonant laser standing wave analytically and numerically. The periodic orbits of the perturbed system whose initial conditions are near the unperturbed heteroclinic orbit have been constructed. The necessary and sufficient conditions determining the stability of the orbits have also been established. These conditions contain the Melnikov criterion for the onset of chaos and describe the intersections of the stable and unstable orbits. The stable periodic orbits which depend on these conditions possess the sensitivity to initial conditions and are embedded in the chaotic attractor. The numerical results on the basis of stability conditions showed that

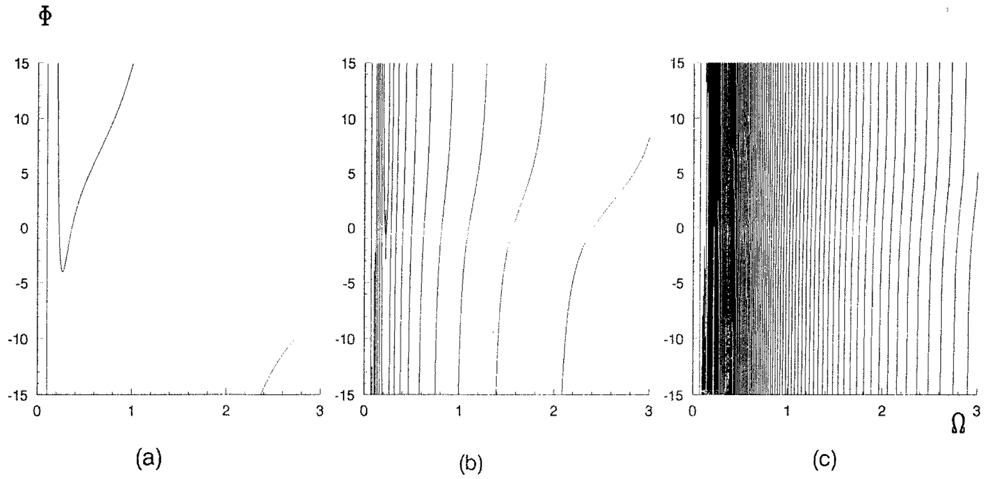

Figure 1. Plot of the control parameter ϕ versus Ω from $F_{1+} - F_{1-} = \Delta(\tau_0) = 0$ for (*a*) $C = 1$, (*b*) $C = 10$ and (*c*) $C = 100$.

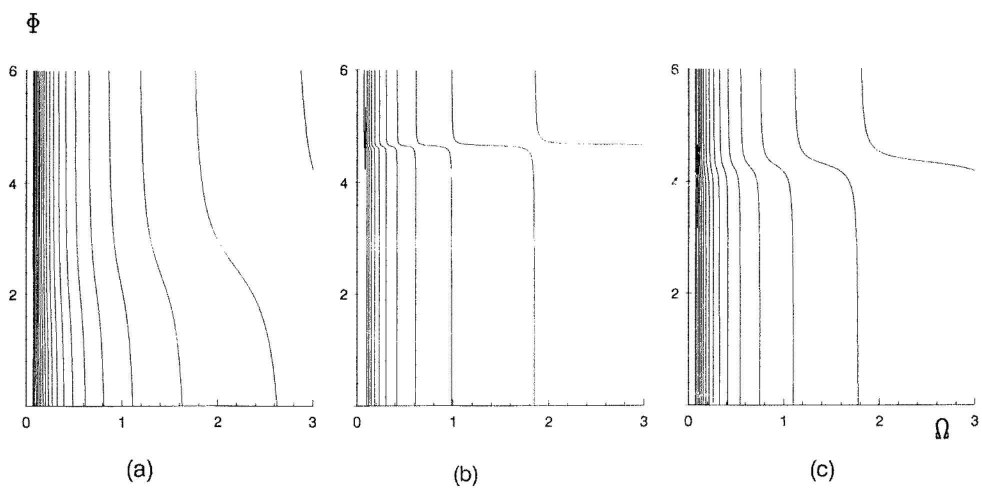

Figure 2. Control curves from $F_{1+} = 0$ in equations (17) for $C = 10$ with (*a*) $A_1 = 0$, (*b*) $A_1 = 2$ and (*c*) $A_1 = 4$.

for any initial constants A_1 and great C chaos can be controlled through adjustments of the control parameters Ω and ϕ. Such stable orbits will be useful for practicable problems.

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